Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 15, Issue. 1, No.3 : 2024
ISSN : **1906-9685**

STRONG T-COLORING OF GRAPHS AND IT'S SIGNIFICANCE

Mrs.M.Kalyani, Assistant Professor, Department of Mathematics, Excel College for Commerce and Science, Komarapalayam.Email: kalyanimaths02@gmail.com

ABSTRACT

Graph colouring is general. Positive integer graph and finite set T. A graph a-coloring assigns a colour to each element in T so that two elements connected by an edge have different colours. For more colour options, "Strong-coloring" (S-coloring) is used. We have a graph with finite positive numbers. An S-coloring of the graph assigns a colour to each element in the set, thus elements connected by edges are assigned distinct colours, and any two edges are assigned different colours. S-chromatic number is the minimum number of colours needed to S-color a graph. The "S-span" is the biggest graph colouring value after inspecting all vertex pairs. For all graph colorings, "ST-coloring c of G" returns the lowest "S-span" value. An "edge span" is the greatest value of all edges in a graph G, while a "S-edge span" is the lowest value from all ST-colorings of G. T-coloring is a generalised colouring of a graph $G = (V, E)$. The graph $G = (V, E)$ and a finite set of positive numbers, including 0, are shown. The T-coloring of G is a function $f : V(G) \rightarrow Z+ \cup \{0\}$, where if uw is an edge in $E(G)$, the absolute difference between $f(u)$ and f(w) is not The term "Strong T-coloring" encompasses more than just that. S-coloring of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow Z+ \cup \{0\}$. If uw $\in E(G)$, then $|f(u) - f(w)|$ ST-coloring requires a minimum of 3 colours, which is G's ST-Chromatic number. The ST-span of a graph G is the largest colour difference between any two vertices using ST colouring c. G has the lowest spc(G) of all ST colorings c. The largest absolute difference between $c(u)$ and $c(v)$ values for all edges in G is the cST-edgespan $espc$ (G The minimum S-edge span $espST(G)$ is determined by considering all possible ST-colorings.

Keywords: T-coloring, ST-coloring, span, edge span.

INTRODUCTION

The graphs analysed in this study are finite, simple, and undirected. For definitions not addressed in this work, one can consult. In T-coloring, the vertices represent transmitters, and an edge exists between two transmitters if they interfere with each other. In that particular model, a single fixed value of T is used to account for all the interference. A T-coloring of a graph $G = (V, E)$ is an extension of graph colouring. Consider a set T that is strictly less than the union of sets Z and ∪, excluding the element 0. This set T is assumed to be fixed. A graph G is said to have a T-coloring if there exists a function $f : V(G) \rightarrow Z+ \cup \{0\}$, such that for every edge $uw \in E(G)$, the absolute difference between $f(u)$ and $f(w)$ is not an element of T. To obtain additional results on T-coloring, one may refer to the poll on T-coloring as mentioned in. A T-coloring of a graph provides a useful representation for illustrating the interference that occurs between transmitters. Assume that this fixed set T can change for each interfering transmitter. Under these circumstances, it is possible to represent the scenario using a novel concept called strong T-coloring of G, which serves as an extension of T-coloring for a graph. Consider a graph G and a finite set T consisting of non-negative integers. An ST-coloring of graph G is a function f: V(G) → Z+ ∪ {0} such that for any u, w in V(G) [1-11].

(i) uw ∈ E(G) then |f(u) − f(w)| Ø T and

(ii) |f(u) − f(w)| G |f(x) − f(y)| for any two distinct edges uw, xy in E(G).

The ST-Chromatic number of G is the minimum number of colors needed for a ST coloring of G and it is denoted χST (G) .

The following observation is immediate.

Observation 1: (i) $3ST\ (G) \geq 3(G) = 3T$

Theorem 2.1. If (G)

G is a simple connected graph then there exists a ST-coloring.

Proof.: Let G be a graph with $V\ (G) = \{v1,\ v2,\ .\ .\ .\ ,\ vn\}$ and let T be the set of positive integers containing 0 with k as its largest element. Define ST -coloring of G as follows.

$c(v\text{i}) = (k + 2)n{+}\text{i}$ for $1\ \leq\ \text{i}\ \leq\ n$

Now we need to prove that

$|f(v\text{i}) - f(v\text{j})|$ G $|f(vl) - f(vm)|$ (1)

where $v\text{i}\ v\text{j},\ vl\ vm \in E(G)$.

If $v\text{i}\ v\text{j}$ and $vl\ vm$ are adjacent then clearly equation (1) holds. Hence, assume $v\text{i}\ v\text{j}$ and $vlvm$ be two non adjacent edges. Therefore i, j, l, $m$are distinct positive integers. W.l.g assume that i is the largest integer and m is the least integer. Then either $m \leq \text{j} \leq l \leq \text{i}$ or $m \leq l \leq \text{j} \leq \text{i}$.

Case(i) : $m \leq \text{j}\ \leq\ l\ \leq\ \text{i}$.

Suppose (1) is not true. Then we have,

$|f(v\text{i}) - f(v\text{j})| =\ |f(vl) -\ f(vm)|$

$(k + 2)n{+}\text{i} - (k + 2)n{+}\text{j} = (k + 2)n{+}l - (k + 2)n{+}m\ (k + 2)\text{i} - (k + 2)\text{j} = (k + 2)l - (k + 2)m$

$(k + 2)\text{i}{-}m - (k + 2)\text{j}{-}m =\ (k + 2)l{-}m - 1$

$(k + 2)\text{i}{-}m + 1\ =\ (k + 2)l{-}m + (k + 2)\text{j}{-}m$ (2) Therefore $(k + 2)\text{i}{-}m$ and $(k + 2)l{-}m + (k + 2)\text{j}{-}m$are consecutive integers, where $\text{i} - m > l - m > \text{j} - m > 2$.

Let $k + 2 = a$, $\text{i} - m = x$, $l - m = y$, $\text{j} - m = z$.

$ay + az\ -\ ax\ \leq\ ax{-}1\ +\ ax{-}2\ -\ ax$

$\leq ax{-}2(a + 1 - a2) \leq 0$

Which in turn implies that $(k + 2)l{-}m + (k + 2)\text{j}{-}m - (k + 2)\text{i}{-}m$ is negative, a contradiction to (2). Similar is the case when $m \leq l \leq \text{j} \leq \text{i}$.

Theorem 2.2. Let T be any set. If H is the subgraph of a graph G then $3ST\ (H) \leq 3ST\ (G)$.

We prove that the strong chromatic index for each k-degenerate graph with maximum degree Δ is at most (4k − 2)Δ − k(2k − 1) + 1. A strong edge-coloring of a graph G is an edge-coloring so that no edge can be adjacent to two edges with the same color. So in a strong edge-coloring, every color class gives an induced matching.

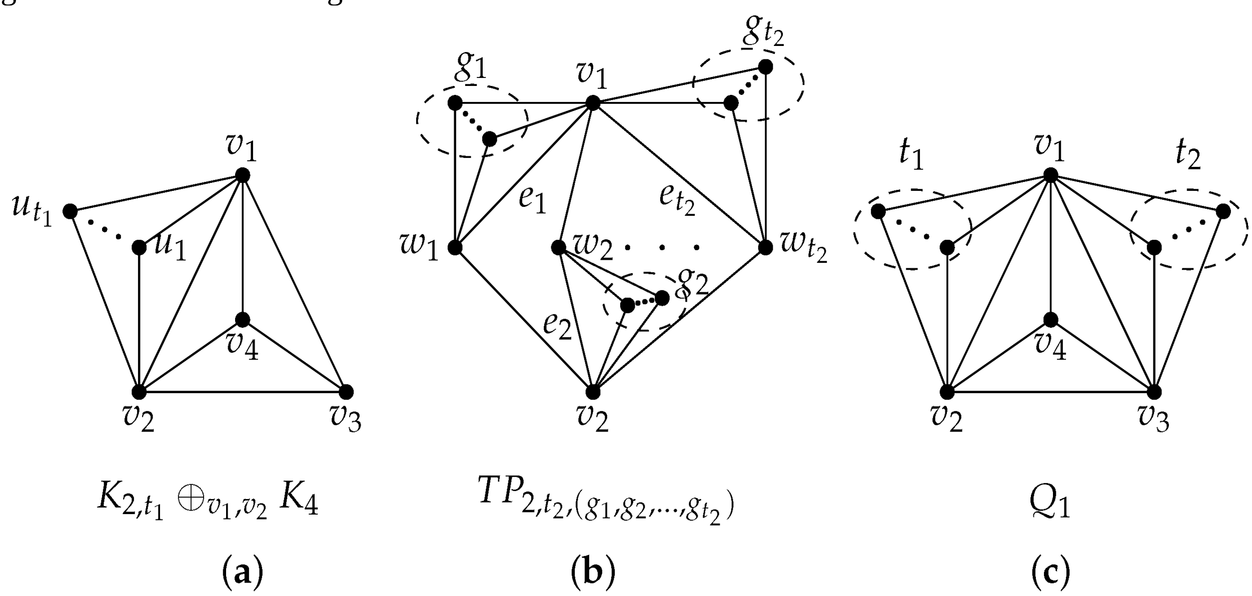

Figure 1: Strong Edge Coloring

The strong chromatic index χ′ (G) represents the smallest possible number of colours required to strongly colour the edges of graph G. Fouquet and Jolivet introduced this concept. Erd˝os and Neˇsetˇril presented several open questions during a seminar in Prague in 1985. One of these difficulties is referred to as Conjecture 1 (Erd˝os and Neˇsetˇril, 1985). If G is a simple graph with a maximum degree Δ, then the chromatic index χ′ (G) is at most 5Δ2/4 if Δ is even, and at most (5Δ2 − 2Δ + 1)/4 if Δ is odd. This conjecture holds true for triangles with a maximum of three sides. Cranston demonstrated that the chromatic index of graph G, denoted as χ′ (G), is at most 22 when the maximum degree of G, denoted as Δ, is equal to 4. Chung, Gy´arf˝as, Trotter, and Tuza demonstrated that the top limits correspond precisely to the number of edges in graphs that do not contain a subgraph isomorphic to 2K2. Molloy and Reed demonstrated that graphs with a sufficiently big maximum degree Δ have a strong chromatic index that is at most 1.998Δ2. A graph is considered k-degenerate if the lowest degree of every subgraph is no greater than k. In a recent study, Chang and Narayanan (2012) demonstrated that a graph with a maximum degree of Δ and a 2-degenerate property has a strong chromatic index that is at most 10Δ − 10. Luo and the author enhanced the maximum limit to 8Δ − 4.

The following conjecture, known as Conjecture 2 (Chang and Narayanan), was proposed. There is a fixed constant, denoted as c, such that for any k-degenerate graphs G with a maximum degree Δ, the chromatic index χ′ (G) is less than or equal to ck2Δ. Additionally, the k2 can be substituted with k. This study presents a more robust version of the conjecture, which we demonstrate. In contrast to the priming methods, we discover a distinct arrangement of the edges and achieve the subsequent outcome by employing a greedy colouring technique.

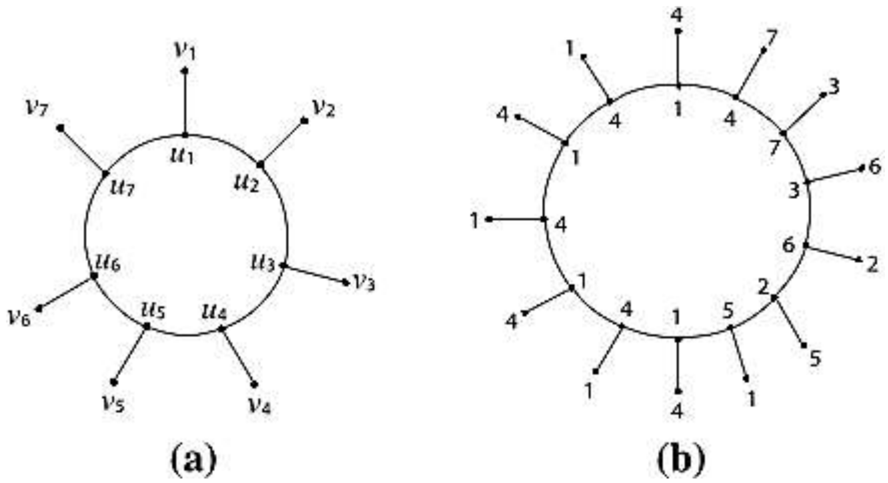

Figure 2: T-Coloring of Certain Networks

Theorem 1: The strong chromatic index of a k-degenerate graph with maximum degree Δ is at most (4k − 2)Δ − k(2k − 1) + 1. Therefore, graphs that are 2-degenerate have a high chromatic index that is at most 6Δ − 5. Evidence. According to the definition of k-degenerate graphs, when we remove all vertices with a degree of k or less, the resulting graph either has no edges or has new vertices with a degree of k or less. This simple fact about k-degenerate graphs may also be found in reference [2]. Consider a graph G that is k-degenerate. There is a vertex u in the graph G such that u is connected to at most k vertices with a degree greater than k. Furthermore, if the graph G has a degree sum greater than k, then it is possible to select a vertex u with a degree exceeding k [12-16].

A vertex u is classified as a special vertex if it is connected to no more than k vertices with a degree greater than k. A special edge is defined as an edge that is connected to both a special vertex and a vertex with a degree of at most k. Consequently, it can be inferred that every k-degenerate graph possesses a distinct edge, and if the maximum degree (Δ) is less than or equal to k, then each vertex and edge are unique. The ordering of the edges of G is as follows. Initially, we identify a distinctive edge in G, place it at the start of the list, and subsequently eliminate it from G. Reiterate the aforementioned step in the remaining graph. Upon completion of the process, we obtain a sequentially arranged collection of the edges in G, denoted as $e_1, e_2, \ldots, e_m$, where m equals the cardinality of E(G). Em is the unique edge that we initially selected and added to the list.

Consider the graph G_i, which is formed from the first i edges in the list, where i ranges from 1 to m. Therefore, e_i is an exceptional edge in G_i. We are currently enumerating the edges of graph G_i that are within a distance of one from vertex e_i in graph G. The edges in G_i can be referred to as blue edges, while the edges in $G - G_i$ can be referred to as yellow edges. Consider the endpoints of edge e_i as u_i and v_i, where u_i is a designated vertex in graph G_i. Initially, we tally the number of blue edges that are connected to the vertex u_i and its adjacent vertices. The vertex u_i has three types of neighbours: the neighbours in X_1 who share blue edges with u_i and have a degree greater than k, the neighbours in X_2 who share blue edges with u_i and have a degree of at most k (therefore $v_i \in X_2$), and the neighbours in X_3 who share yellow edges with u_i. According to the definition, the absolute value of X_1 is less than or equal to k. Therefore, there may be at most $|X_1|\Delta + k(|X_2| - 1)$ blue edges that are connected to $X_1 \cup (X_2 - \{v_i\})$. For every vertex u in X_3, the edge uu_i is coloured yellow in G_i, but it will become a unique edge in G_j for some j that is greater than i. Either u or u_i has a degree of at most k in G_j (and hence also in G_i). If u_i has a degree of at least k in G_m for some m, then all yellow edges connected to u_i in G_m must have a degree of at most k - 1 in G_m, in order for the yellow edges to be considered exceptional later. Among the vertices in X_3, the maximum number of vertices (x) that can have a degree greater than k in G_i is given by the formula $x = \max\{0, k - |X_1| - |X_2|\}$. All other vertices in X_3 have a degree at most $k - 1$ in G_i. Hence, the maximum number of blue edges incident to X_3 is $x\Delta+(|X_3|- x)(k - 1)$. Given the conditions that $d(u_i)$ is less than or equal to Δ, $|X_2|$ is less than or equal to Δ, and $|X_1| + x$ is less than or equal to k, we may conclude that the maximum number of blue edges within distance one to e_i from the u_i side (excluding the edges incident to v_i) is $(|X_1|+x)\Delta+(k - 1)(d(u_i)-|X_1|- x- 1) +|X_2|- 1$, which is also less than or equal to $2k\Delta- k^2$.

In addition, we tally the number of blue edges that are connected to vertex v_i and its adjacent vertices. Similarly, each vertex v_i has two types of neighbours: the neighbours in Y_1 that share blue edges with v_i, and the neighbours in Y_2 that share yellow edges with v_i. Given that e_i is a unique edge, the cardinality of Y_1, denoted as $|Y_1|$, is less than or equal to k. Therefore, the maximum number of blue edges incident to $Y_1 - \{u_i\}$ is $(|Y_1| - 1)\Delta$. For every vertex v in Y_2, vv_i represents a yellow edge in G_i, but it will become a unique edge in G_s for some s greater than i. Similarly to the previous statement, in G_i, there can be at most $k - |Y_1|$ vertices in Y_2 that have a degree greater than k. All other vertices in Y_2 have a degree of at most k - 1 in G_i. The maximum number of blued edges occurring to Y_2 is given by the expression $(k - |Y_1|)(\Delta - 1) + (|Y_2| - (k - |Y_1|))(k - 1)$.

The maximum value of the expression is $(|Y_1| - 1)\Delta + (k - |Y_1|)(\Delta - 1) + (|Y_2| - (k - |Y_1|))(k - 1)$, which is less than or equal to $(2k - 2)$.The expression $\Delta - k(k - 1)$ represents the difference between the value of Δ and the product of k and (k - 1). In G_i, the maximum number of blue edges at a distance of one from e_i is 2k.The inequality $\Delta - k^2 + (2k - 2)\Delta - k(k - 1)$ is less than or equal to $(4k - 2)\Delta - k(2k - 1)$. Proceed to colour each edge in the list individually using a greedy approach. For each i, when it is the time to colour e_i, just the edges in G_i (the blue edges) have been coloured. Given that there are a minimum of $(4k - 2)\Delta - k(2k - 1) + 1$ colours available, we can assign colours to the edges in such a way that edges within a distance of one have distinct colours.

The ST-SPAN AND The ST-Edge Span

Consider c as a proper vertex colouring of G using ST colours. If k represents the highest colour assigned to a vertex of G using the ST-coloring c, then the colouring $\bar{c}$ of G, defined as $\bar{c}(v) = k + 1 - c(v)$ for each vertex v of G, is likewise a ST-coloring of G. This colouring is referred to as the complementary colouring of c.

Let's define the ST-coloring c of a graph G as a colouring scheme where each vertex is assigned a colour. The cST-span spc(G) is the largest difference in colour between any two vertices u and v in G. The ST-span spST(G) is the smallest value of spc(G) obtained by considering all possible ST-colorings c of G. The cST-edgespan espc(G) represents the highest value of $|c(u) - c(v)|$ among all edges uv in G. On the other hand, the ST-edge span espST(G) is defined as the smallest value of espc(G) when considering all possible ST-colorings of G.

Every graph G has an ST-coloring in which one vertex is assigned the colour 0. If c' is an ST-coloring of a graph G, where $t \geq 1$ is the smallest colour assigned to any vertex of G, then the colouring c of G defined by $c(v) = c(v') - a$ for each $v \in V(G)$ is also an ST-coloring of G. In this new colouring, there is a vertex assigned the colour 0 by c, and the cST-span of G is the same as the c′ST-span of G. Therefore, for a given finite collection of non-negative integers, spST(G) is defined as the minimum of the highest $c(v)$ value, where the maximum is calculated for all vertices v in G, and the minimum is calculated for all ST-colorings of G. If spST(G) = l, then there exists an ST-coloring $c : V(G) \rightarrow \{0,1,2, . . . , l\}$ of G, where at least one vertex is coloured 0 and at least one vertex is coloured l. For every graph G, it holds that $3ST(G)$ is less than or equal to spST(G) [17-19].

Theorem 3.1. For all graphs G,

(i)spT(G) ≤ spST(G) , (ii)$e$spT(G) ≤ espST(G)

Proof: Let T be any finite set of non negative integers containing 0. Every ST-coloring of G is also a T-coloring of G. Hence, spT(G) ≤ spST(G), $e$spT(G) ≤ espST(G). Theorem 3.2. Let H be a subgraph of a graph G. For each finite set T of nonnegative integers containing 0,

(i) spST(H) ≤ spST(G) (ii)espST(H) ≤ $e$spST(G)

Proof is similar that of (i).

Corollary 3.1. If G is weakly γ-perfect then spST(G) = espST(G) = spST(Kk)

In closing this paper, we mention some most important questions which remain.

Conjecture: Let T be a finite set of non negative integers containing 0. If G is a graph with $3ST(G) = k$ and $\omega(G) = l$, then spST(Kl) ≤ $e$spST(G) ≤ spST(G) ≤ spST(Kk).

Open problems:

(i) For certain families of graphs, determine $3ST(G)$.

(ii) For which graph , $3ST(G) = |V(G)|$?

(iii) Find the values of spST(Kn) and $e$spST(Kn) when T is a k-initial set.

(iv) For certain families of graphs, compute spST(G) and $e$spST(G).

CONCLUSION

There are two types of graph colouring problems: T-coloring and Strong T-coloring. These graph colouring problems are generalisations of the channel assignment problems typically encountered in broadcast networks.During this presentation, we will discuss the concept of distance graphs as a tool for analysing the entire ST-coloring problem, and we will also investigate the complexity of this subject matter. Obtaining a deeper comprehension of the ST-coloring problem might be possible with the application of additional analysis to the structure of distance graphs.

REFERENCES

1. Balakrishnan.R, Ranganathan.K; A Text Book of Graph Theory; Springer, 2000.
2. Cozzens.M.B. and Roberts.F.S., T-Coloring of graphs and the channel assignment problem, Congressus Numerantium, 35 (1982), 191-208.
3. D. W. Cranston, Strong edge-coloring of graphs with maximum degree 4 using 22 colors. Discrete Math. 306 (2006), no. 21, 2772–2778.

4. J. L. Fouquet and J. L. Jolivet, Strong edge-coloring of graphs and applications to multi-k-gons. Ars Combinatoria, 16A:141–150, 1983.
5. P. Hor´ak, Q. He, W. T. Trotter, Induced matchings in cubic graphs. J. Graph Theory 17 (1993), no. 2, 151–160.
6. R. Luo and G. Yu, A note on strong edge-colorings of 2-degenerate graphs, http://arxiv.org/abs/1212.6092.
7. M. Molloy, B. Reed, A bound on the strong chromatic index of a graph. J. Combin. Theory Ser. B 69 (1997), no. 2, 103–109.
8. M. Steibitz, D. Scheide, B. Toft, and L. M. Favrholdt, Graph Edge Coloring, A John Wiley & Sons, Inc, 2012.
9. Gary Chartrand, Ping Zhang , Chromatic graph theory , Discrete Mathematics and its applications, CRC Press, Taylor and Francis Group,2009.
10. Hale.W.K., Frequency assignment: Theory and applications, Proceedings of the IEEE; 68(1980), 1497-1514.
11. Jai Roselin.S, Benedict Michael Raj.L ., T-Coloring of certain non perfect Graphs,Journal of Applied Science and Computations,Vol VI, Issue II, February/2019
12. Justie Su-Tzu Juan, *I-fan Sun and Pin-Xian Wu, T-Coloring on Folded Hypercubes, Taiwanese journal of mathematics, 13(4) (2009), 1331-1341.
13. Robert A Murphey, Panos M Paradalos, Mauricio GC Resende, Frequency Assignments Problems, Handbook of Combinatorial optimization, 295-377, 1999.
14. Raychaudhuri.A., Further results on T-Coloring and Frequency assignment problems, SIAM J. Disc. Math., 7(1994), 605-613.
15. Liu.D.D.F., T-colorings of graphs, Discrete Mathematics, 101(1992), 203-212.
16. Tesman.B.A., List T-colorings of graphs, Discrete Applied Mathematics, 45(1993), 277-289.
17. L. D. Andersen, The strong chromatic index of a cubic graph is at most 10. Topological, algebraical and combinatorial structures. Frol´ık's memorial volume. Discrete Math. 108 (1992), no. 1-3, 231–252.
18. G.J. Chang and N. Narayanan, Strong Chromatic Index of 2-Degenerate Graphs, J. Graph Theory, DOI: 10.1002/jgt.21646.
19. F. R. K. Chung, A. Gy´arf´as, W. T. Trotter, and Z. Tuza, The maximum number of edges in 2K2-free graphs of bounded degree. Discrete Mathematics, 81(2):129–135, 1990.